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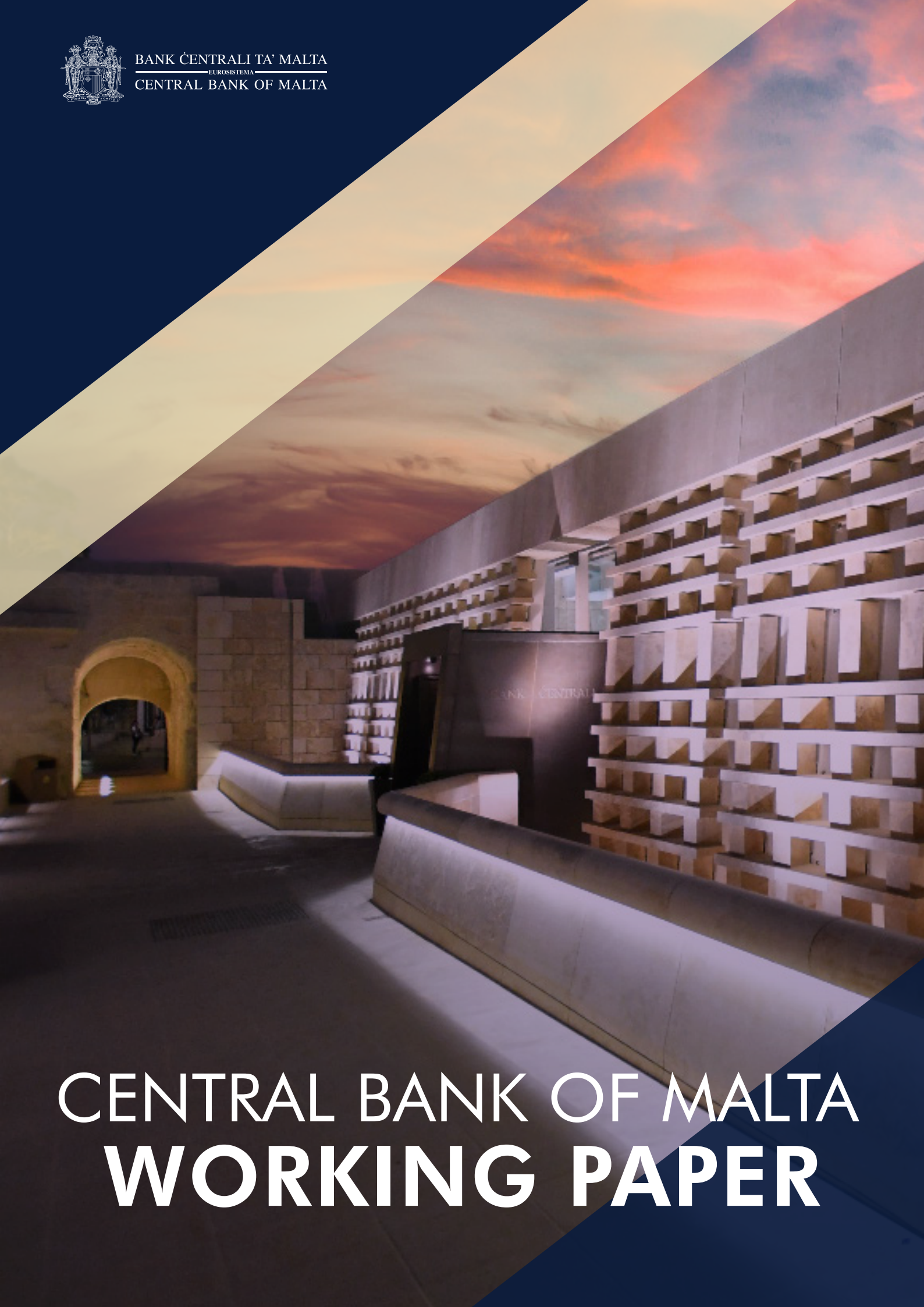

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Wealth inequality and the distributional effects of maximum loan-to-value ratio policy*

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Abstract

Macroprudential policy improves economic outcomes by reducing the likelihood and severity of financial crises. Yet it is pertinent to ask, are there unintended long run consequences to the introduction of a macroprudential policy regime, and are these consequences conditional on the a priori level of wealth inequality? This paper answers these questions by looking at the effect of a reduction in the maximum loan-to-value (LTV) ratio on homeownership rates, house prices and housing wealth inequality across two economies with different initial wealth dispersion. It uses a heterogeneous agent model in which households face uninsurable income risk and an endogenous borrowing limit in the form of a collateral constraint. This constraint is initially loose, allowing households to lever up against the collateral value of their housing. A reduction in the LTV limit tightens the borrowing constraint, and lowers homeownership as a greater share of households no longer afford the downpayment. The key finding of this paper is that initial conditions matter; the lower is wealth inequality *ex-ante*, the higher is the fall in house prices and the greater is the rise in the share of constrained homeowners and housing wealth inequality *ex-post*. The effects are also non-linear in the LTV ratio, with progressively stronger effects at lower LTV ratios, especially when inequality is comparatively low.

JEL Classification: E21, G11, G28, G51, R21

Keywords: *wealth distribution, housing, macroprudential policy, loan-to-value, saving*

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1 Introduction

Macroprudential policy regimes were introduced in several countries in the wake of the financial crisis, with the aim of stabilizing financial cycles and discourage excessive risk-taking. Maximum loan-to-value (LTV) limits for household credit is one such policy that belongs to this toolkit. An LTV cap limits the leverage that a household can build and encourages saving by requiring households to put up a higher downpayment when purchasing real estate. For example, the Reserve Bank of New Zealand introduced a maximum LTV ratio of 80% for most households, allowing only a 'speed limit' of 10% of transactions to be at a higher LTV ratio.[1] The experience so far seems to have been positive, however it is clear to policymakers that these policies can have unintended consequences (Cassidy and Hallissey, 2016; Lu, 2019). A household is prevented from buying a house if it does not have enough equity, even if it can service the debt. Besides forcing such a household to rent, this policy limits the household's asset portfolio, and can potentially deprive the wealth gains brought about by house price appreciation.

This paper focuses on the long-run unintended consequences of LTV policies, notwithstanding the fact that preventing financial crises is undoubtedly welfare-improving. The discussion on unintended consequences is especially relevant since housing tends to be the most valuable asset held by 'middle class' households (Piketty, 2014, p.260). In the US, for example, wealth of the bottom 90% of households is composed mainly of housing as the main asset, and very few financial holdings, while liabilities constitute primarily of housing debt and other debt, such as student and car loans (Kuhn et al., 2020).[2] Meanwhile, renters account for about a third of all households in the UK and US and tend to have little wealth (Davis and Van Nieuwerburgh, 2015; Cloyne et al., 2020). Cloyne et al. (2019) show that the collateral channel of housing is important for explaining movements in household borrowing in the UK.

The literature on macroprudential policies is growing but most studies focus on short to medium-term business cycle frequencies, while the long term effects of such policies are less explored. Baker (2017) argues that the potential for distributional effects stemming from macroprudential policy is high. Frost and van Stralen (2018) look at the empirical link between macroprudential policy tools and measures of income inequality over the period 2000–2016. They find a positive, although not necessarily causal, relationship between the use of the LTV ratio and other macroprudential policies and income inequality. However, results for advanced economies are mostly not statistically significant, likely due to their relatively limited experience with LTV policy prior to the financial crisis. In emerging and developing economies the use of LTV ratio policy seems to be more strongly associated with higher income inequality. Carpantier et al. (2018) focus instead on wealth inequality. Using data for the euro area, they find that higher LTV ratios are associated with an increase in wealth inequality. These authors also stress that their results are not causal, and note that most households with high LTV mortgages tend to be on the lower part of the wealth distribution.[3]

[1] The speed limit is reviewed and tightened or eased countercyclically, depending on a number of indicators.

[2] This means that the portfolio of the absolute majority of households is highly specialized and, especially for the bottom 50%, also highly leveraged.

[3] Richter et al. (2019) show that the output costs of LTV policies are small and not statistically significant in advanced economies.

The contribution of this paper is to study the long run distributional effects of LTV ratio policies as a function of initial wealth inequality, focusing on changes to housing wealth inequality and homeownership. To answer this question this paper uses a heterogeneous agent model with a housing tenure choice and a collateral constraint. The introduction of a macroprudential policy, reflected in a reduction in the maximum LTV ratio, lowers the homeownership rate, house prices and aggregate leverage. At the same time, it raises the share of homeowners who are up against the borrowing limit, despite the fall in house prices, and housing wealth inequality rises. This is in line with the findings of other studies in the literature. The key finding is that the effects on house prices, leverage and housing wealth inequality are conditional on the initial state; the lower is initial wealth inequality, the stronger the effects. This is because policy tends to have an effect on a bigger share of the population when wealth inequality is lower, since *ex-ante* households are relatively less heterogeneous in their wealth holdings. The same policy applied to two countries with different levels of wealth inequality affects them differently. These findings are robust to the level of the long run interest rate and the initial homeownership rate. These results highlight the importance of relaxing the representative agent assumption in models and instead cater for agents on the entire distribution, especially when using such models for policy.[4]

There are several studies that model movements in the LTV ratio and analyse the associated implications on the housing market and aggregate outcomes. Favilukis et al. (2017) study financial liberalization in the US by simulating an increase in the LTV ratio from 75% to 99%. They find that this causes a reduction in housing wealth inequality in the long run but an increase in financial wealth inequality. However they do not include a rental market in their model, and this could have important implications for the results. Kiyotaki et al. (2011) and Sommer et al. (2013) also study the effects of a relaxation in the collateral constraint in a model in which households can choose to either rent or buy a house. They show that lowering the downpayment requirement (raising the LTV) increases homeownership as it enables renters to climb the housing ladder through a highly leveraged house purchase. However, they find a small effect on aggregate rents and house prices, attributed to the fact that the share of wealth of such households in the economy is low. These studies do not look at inequality measures before and after the relaxation of borrowing limits.[5] Similarly, Iacoviello and Pavan (2013) study the effects of an increase in the LTV ratio on the rate of homeownership and household indebtedness. They find that the increase in the LTV ratio increases homeownership and aggregate debt relative to output. However, this shock is neutral on wealth inequality since the authors assume that house prices are exogenous and therefore do not respond to the increase in housing demand.[6]

Although several countries have introduced similar macroprudential policies over the past

[4]In recent work, Amberg et al. (2021) show that monetary policy shocks in Sweden affect households across the entire income distribution, with considerable heterogeneity.

[5]Kiyotaki et al. (2011) also show that the result carries through when the collateral constraint is tightened completely, requiring all housing to be purchased using own resources. In this case, the resulting effect on aggregate prices and quantities is also limited, even though about half of households become renters. The risk-free interest in these two models is exogenous and does not adjust.

[6]Other noteworthy studies on this topic are Ortalo-Magne and Rady (2006) and Kaplan et al. (2020). A summary of earlier work can be found in Jeske (2005). Other studies which study distributional implications of different policies on the housing market are Gervais (2002), Jeske et al. (2013), Floetotto et al. (2016), Sommer and Sullivan (2018) and İmrohoroğlu et al. (2018).

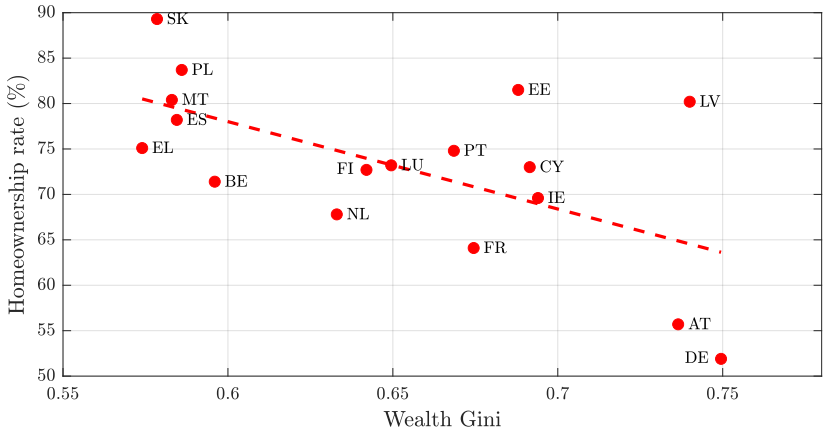

Figure 1: Wealth inequality and homeownership rates in selected European countries

Notes: Data is for the year 2015. EL – Greece, SK – Slovakia, MT – Malta, ES – Spain, PL – Poland, BE – Belgium, NL – Netherlands, FI – Finland, LU – Luxembourg, PT – Portugal, FR – France, EE – Estonia, CY – Cyprus, IE – Ireland, AT – Austria, LV – Latvia, DE – Germany. Source: Eurostat (experimental statistics and EU-SILC). The dashed red line in panel (b) denotes the line of best fit based on a linear OLS regression ($R^2 = 0.37$), for illustration purposes.

years, it is not clear whether the expected long-run effect of these policies is homogeneous. This is because countries are heterogeneous along several dimensions, and differ *inter alia* in wealth inequality and homeownership rates. For example, wealth Gini coefficients range from less than 0.6 to more than 0.7 across several European countries (Figure 1), while homeownership rates vary from under 50% to over 80%. Moreover, 19 European countries had an LTV cap by mid-2018, most varying between 60–95% (Arena et al., 2020). It is therefore important to understand to what extent heterogeneity in initial conditions matters for the long run unintended consequences of such policies.

At this stage two caveats are in order. First, a complete discussion of macroprudential policy would weight the costs of action with the costs of inaction. This paper however does not take a position on optimal policy, as it completely sidesteps the benefits brought about from greater financial stability. As a result, any welfare implications derived from the results of the analysis that follows would miss an important element that motivates the implementation of such a policy in the first place. It is undisputed that the existence of a policy that prevents overborrowing and excessive leverage, and therefore borrower default and fire sales, is welfare-improving. Therefore, this paper takes policy as given and analyses how it affects the housing market by reallocating portfolios. Second, a macroprudential policy regime generally leverages several tools simultaneously, such as borrower debt to income and bank capital to asset ratio limits. Furthermore, policymakers typically adopt these tools differently across borrower types, for example by distinguishing between first-time buyers and buy-to-let investors.[7] In this paper the focus is specifically on the LTV ratio, to be able to study the distributive effects of policy derived solely from this tool, as well as to keep the analysis close to the literature cited. Moreover,

[7]See Privitelli (2019) and Gatt (2023) for case studies related to Malta, and New Zealand and Ireland, respectively.

since households are infinitely lived in the model, it is not possible to discriminate among first-time buyers, as in practice. The rest of the paper is organised as follows. Section 2 goes through the model and section 3 characterises the initial stationary equilibrium. Section 4 reports simulation results following a tightening of the LTV ratio and section 5 discusses sensitivity analysis. Section 6 concludes.

2 The model

I use a framework similar to Gervais (2002), Silos (2007), Kiyotaki et al. (2011) and Iacoviello and Pavan (2013), although I abstract from life-cycle profiles as in Carroll and Dunn (1997) and Guerrieri and Lorenzoni (2017).[8] In the presence of borrowing constraints, a share of households finds it optimal to rent rather than own a house. To preview results, renters will be wealth-poor households while owners represent the wealthier cohort of society, consistent with the data.

2.1 Households

Time is continuous and the economy is populated by a continuum of infinitely lived dynastic households of mass one, which receive stochastic income endowments which transition between two states $y_j \in \{y_1, y_2\}$. There is no aggregate uncertainty. In the absence of complete asset markets, households are unable to perfectly insure against this risk, so their earnings fluctuate over time. Households have Constant Relative Risk Aversion preferences over total consumption C, which is a Cobb-Douglas aggregate of non-durable consumption goods c and housing services s, and discount the future at the rate ρ. A household derives housing services by either owning a house of size h or by renting one; however, renting incurs a utility loss $\psi > 0$, representing the tenant's limited control over the asset.[9] I assume housing is fully divisible and freely adjustable, as in Kiyotaki et al. (2011) and Jeske et al. (2013).

A house yields a service that is linearly related to its size ($s = h$). To simplify notation I do not index households by a subscript. Household utility is given by:

$$u(c, s, \mathbf{1}_{\text{rent}}) = \frac{1}{1-\sigma}\left[\left(\frac{c_t}{\alpha}\right)^{\alpha}\left(\frac{(1-\psi\mathbf{1}_{\text{rent}})s_t}{1-\alpha}\right)^{1-\alpha}\right]^{1-\sigma} \tag{1}$$

where the indicator function $\mathbf{1}_{\text{rent}}$ takes a value of one when a household is renting and zero otherwise. Non-seperable preferences across consumption goods and housing services imply that households maintain constant expenditure shares α and $1-\alpha$ respectively across these two components. This follows the preference structure commonly used in the literature, as it simplifies the solution of the model by seperating the intertemporal consumption-saving problem over the aggregate consumption bundle C from the intratemporal allocations over non-durable

[8] The framework is general and has also been used by Chambers et al. (2009), Cho and Francis (2011), Berger et al. (2018) and İmrohoroğlu et al. (2018).

[9] This follows Kiyotaki et al. (2011), Iacoviello and Pavan (2013) and Floetotto et al. (2016), who use this parameter to pin down the equilibrium homeownership rate.

consumption c and durable goods s.[10] Besides housing, households have access to a riskless liquid asset $b > 0$ yielding a return r.[11] A household with $b < 0$ is in debt, and r is then the interest on such debt. Only households which own a house can borrow. Current and prospective homeowners face a borrowing limit, in the form of a collateral constraint over the value of the durable asset (housing), priced in q units relative to the consumption good price, and is governed by a LTV ratio $\theta \in (0, 1)$:

$$-b_t \leq \theta q h_t.$$

The LTV ratio is common to all households. An individual household's assets evolve as

$$db_t + (1 - \mathbf{1}_{\text{rent}}) q dh_t = (y_j + r b_t - \mathbf{1}_{\text{rent}} p s_t - c_t) dt$$

where p is the rental price. Income endowments $\{y_1, y_2\}$ are indexed by j and follow a two-state Poisson processes with intensities $\lambda_j \in \{\lambda_1, \lambda_2\}$ respectively.[12] Households convert income into a consumption good costlessly using linear technology, and therefore the endowments $\{y_1, y_2\}$ can also be thought of as productivities. Prices are determined in equilibrium and households take these as given. Households cannot default on their debt and the borrowing constraint implies that they cannot have net liabilities ($b + qh \geq 0$).

In this paper I focus on the stationary equilibrium of the model. To reduce the dimensionality of the problem it is useful to work in terms of wealth. Following Achdou et al. (2022) and Fagereng et al. (2019), I define household wealth as $W = qh + b$. Since in a stationary equilibrium aggregate prices are constant, the evolution of wealth is given by $dW = qdh + db$. The constraints conditional on income can therefore be re-written as:

$$dW = (y_j + rW - (1 - \mathbf{1}_{\text{rent}}) r q h_j - \mathbf{1}_{\text{rent}} p s_j - c_j) dt \tag{2}$$

$$W \geq \begin{cases} 0 & \text{if renter} \\ (1 - \theta) q h & \text{if owner.} \end{cases} \tag{3}$$

The modified collateral constraint (3) states that the minimum wealth that households have to hold to become or remain homeowners needs to be enough to cover the downpayment $(1 - \theta)$ on housing. To anticipate results, this constraint will lead to a density of agents who are up against the borrowing constraint on the wealth distribution. Meanwhile, renters store their wealth solely through liquid assets.

Similar to the discussion in Gervais (2002), there are financial intermediaries in the background which take deposits from some households and issue loans to others and hold the unowned stock of housing, which they rent out at the rental rate p. There are no frictions in this process. As a result, a no-arbitrage condition in the housing market holds such that in equilibrium the

[10] See, *inter alia*, Kiyotaki et al. (2011) and Sommer et al. (2013).

[11] Therefore, the interest rate on liquid assets and debt is the same.

[12] This is a continuous time jump process, analogous to a two-state Markov process in discrete time. Income jumps from the bad to the good state with Poisson flow intensity λ_1 and from the good to the bad state with intensity λ_2.

rental rate is equal to the opportunity cost of buying a house:[13]

$$p = rq. \tag{4}$$

Households must decide whether to rent or buy a house, a decision that involves a trade-off. Owning incurs a higher utility benefit for a given level of housing services but, when households have little wealth, they might be up against their borrowing constraint, limiting the size of the house that they can buy. On the other hand, renters face no additional constraints on the size of the house to live in, yet renting incurs a utility loss. Denote by $V^o(W)$ and $V^r(W)$ the lifetime value of owning and renting, respectively. The problem of a given household is to choose its housing tenure at every level of wealth for a given income level:

$$\max_{\{s,h\}} \left\{ V_j^o(W), V_j^r(W) \right\},\ j = 1, 2. \tag{5}$$

This problem can be cast as an *optimal stopping time problem* (Stokey, 2009). Assume a given household is an owner. It chooses optimal consumption, the size of the house to own and the stopping time τ at which it switches from owning to renting. An optimal stopping time translates to a wealth threshold W^o below which it is optimal to rent rather than own. The value function of an owner is:

$$V_j^o(W) = \max_{c,s,\tau} \mathbb{E}_t \left\{ \int_0^\tau e^{-\rho t} u(c, s|_{\mathbf{1}_{\text{rent}}=0}) dt + e^{-\rho\tau} V_j^r(W) \right\} \tag{6}$$

subject to the constraints (2)-(3). The household problem separates into a dynamic intertemporal consumption-saving problem and a static intratemporal problem over non-durable and durable goods for a given level of total expenditure. It is convenient to discuss these separately, starting with the dynamic problem.

Denote by P the aggregate price index corresponding to the consumption bundle C, then total expenditure for a given income realisation j is given by:

$$PC_j = c_j + rqs_j. \tag{7}$$

Note that by the no-arbitrage condition (4), it is irrelevant for total expenditure whether the household is a renter or an owner, since the cost of a unit of housing services is the same in equilibrium.[14] The intertemporal problem is solved using a recursive formulation. Following Achdou et al. (2022), Kaplan et al. (2018) and Nuno and Moll (2018), the associated Hamilton-Jacobi-Bellman (henceforth HJB) equation for owning is given by

$$\rho V_j^o(W) = \max_C\ u(C|_{\mathbf{1}_{\text{rent}}=0}) + \frac{dV_j^o(W)}{dW}(y_j + rW - PC_j) + \lambda_j(V_{-j}^o(W) - V_j^o(W)) \tag{8}$$

[13]Since I define the sum of liquid and housing assets in terms of wealth, this expression captures the opportunity cost of buying a house instead of investing in the liquid asset, or the user cost. Furthermore, since in a stationary equilibrium prices are constant, this relation does not include the expected growth of house prices.

[14]Furthermore, one unit of owned housing derives one unit of housing services. On the other hand, it matters for utility whether the household is a renter or borrower.

with the constraint that

$$V_j^o(W) \geq V_j^r(W) \tag{9}$$

where the argument C in the utility function is the Cobb-Douglas aggregate over c and s defined in (1). When constraint (9) holds as an equality it is referred to as the value matching condition, which serves as a boundary condition. There is an HJB for each income state $y_j,\ j = 1, 2$. The notation in the HJB is such that $-j = 2$ when $j = 1$ and vice versa. Therefore, the last term on the right captures the expected change in lifetime utility following a jump to the other income state.[15] The HJB equation associated with renting is given by:

$$\rho V_j^r(W) = \max_C\ u(C|_{\mathbf{1}_{\text{rent}}=1}) + \frac{dV_j^r(W)}{dW}(y_j + rW - PC_j) + \lambda_j(V_{-j}^r(W) - V_j^r(W)). \tag{10}$$

The system can be written as an HJB variational inequality (HJBVI) (Bensoussan and Lions, 1982; Øksendal, 1998):

$$\min\left\{\rho V_j^o(W) - \max_C\ u(C|_{\mathbf{1}_{\text{rent}}=0}) - \frac{dV_j^o(W)}{dW}(y_j + rW - PC_j) - \lambda_j(V_{-j}^o(W) - V_j^o(W)),\right.$$
$$\left. V_j^o(W) - V_j^r(W)\right\} = 0. \tag{11}$$

The solution to this problem yields the policy function for total consumption $C_j(W)$, the resulting saving policy function $S_j(W)$ and a wealth ownership threshold W_j^o. For homeowners, the first order condition for total consumption conditional on a given income realisation is given by:

$$C_j(W)^{-\sigma} = P\frac{dV_j^o(W)}{dW}. \tag{12}$$

Optimal saving is then $S_j(W) = y_j + rW - PC_j(W)$, namely the accumulation of wealth that results from following policy $C_j(W)$. An analogous first order condition holds for renters.

The static problem of a household is to choose the optimal bundle $\{c, s\}$ subject to the constraint (7) for a given level of expenditure. Optimal allocations for owners are:

$$s_j(W) = \min\left\{(1-\alpha)\frac{PC_j(W)}{rq}, \frac{W}{(1-\theta)q}\right\} \tag{13}$$

$$c_j(W) = PC_j(W) - rqs_j(W) \tag{14}$$

where a household can be either on the collateral constraint (3) or have an unconstrained house purchase. Denote the cutoff at which this occurs as W_j^u, with $W_j^u > W_j^o$. On the other hand, renters have unconstrained allocation shares α and $1-\alpha$ of total expenditure for consumption of the non-durable good and housing services, respectively. These conditions define the price of the unconstrained bundle C as:

$$P = p^{1-\alpha}. \tag{15}$$

[15] Derivations are provided in Appendix A. See Achdou et al. (2022) for further details.

2.2 The distribution of households

The distribution of households over wealth for a given endowment j is denoted by $G_j(W)$, and this has a density on wealth $g_j(W)$. The law of motion for the stationary distribution satisfies the Kolmogorov Forward (KF) equation:[16]

$$0 = -\frac{d}{dW}\left[S_j(W)g_j(W)\right] - \lambda_j g_j(W) + \lambda_{-j} g_{-j}(W). \tag{16}$$

The right-hand side states that a density of households moves continuously off (on) a given wealth level conditional on income through saving (dissaving), as well as by jumping to (from) the other income state depending on the Poisson intensities. In the stationary equilibrium the distribution is time-invariant and these forces are equal. Hence the left-hand side, which corresponds to the time derivative of $g_j(W)$, is equal to zero. See Appendix A for more details. The densities are normalized to integrate to 1:

$$\int_0^\infty g_1(W)dW + \int_0^\infty g_2(W)dW = 1. \tag{17}$$

It is useful to define the aggregate share of homeowners who are constrained by the borrowing limit μ^{co}:

$$\mu^{\text{co}} = \frac{\sum_j \int_{W_j^o}^{W_j^u} dG_j(W)}{\sum_j \int_{W_j^o}^{\infty} dG_j(W)} = \frac{1}{1 - G_1(W_1^o) - G_2(W_2^o)} \sum_j \int_{W_j^o}^{W_j^u} dG_j(W) \tag{18}$$

and the share of households who are either renters or constrained owners μ^{rco}:

$$\mu^{\text{rco}} = \sum_j \int_0^{W_j^u} dG_j(W). \tag{19}$$

2.3 Market clearing

I close the model by assuming a fixed aggregate housing supply which is normalized to 1. Equilibrium in the housing market implies all houses are either rented or owned:

$$\sum_{j=1}^{2} \int_0^\infty \mathbf{1}_{\text{rent}} s_j(W)dG_j(W) + \int_0^\infty (1 - \mathbf{1}_{\text{rent}}) s_j(W)dG_j(W) = 1. \tag{20}$$

The interest rate r is fixed and exogenous, and in the background a central bank provides enough liquidity into the economy to maintain this rate.

[16] This is also known as the Fokker-Planck equation; see Achdou et al. (2022) for more details.

2.4 Boundary conditions, equilibrium and solution method

Note that despite the inequality constraint (3) in the optimization problem, the first order condition (12) for households holds as an equality in the interior of the state space.[17] The problem is then subject to a *state constraint boundary condition* when individual wealth falls to the minimum level permissible $\underline{W} = 0$ given by (3):

$$\frac{dV_j(\underline{W})}{dW} \geq P^{\sigma-1} y_j^{-\sigma}. \tag{21}$$

This boundary condition implies that the return to saving at $\underline{W}$, as measured by the change in the value function, is equal to or better than the marginal utility of consuming all available resources. It ensures that any household whose wealth is driven towards $\underline{W} = 0$ through a long spell of low income shocks abides by the non-negativity constraint on wealth and adjusts its consumption.[18]

A stationary equilibrium is defined as a set of prices $\{q, p\}$, value functions $\{V_j^r(W), V_j^o(W)\}$, policy functions $\{c_j(W), s_j(W), C_j(W), S_j(W)\}$ and distributions $G_j(W)$ for each income state j that satisfy the HJBVI (11), the KF equation (16), the market clearing condition (20) and the boundary condition (21). I solve the model numerically on a fine wealth grid $W \in [0, \overline{W})$ with 7,500 points, where $\overline{W}$ is a level of wealth that is high enough such that the density of households at this level is zero. The approach is based on a finite difference method to approximate the derivative of the value function using an upwind scheme, see Achdou et al. (2022) for further details.[19] For a given house price q (and implied rental rate p) and guesses of the value functions for renting $V^r(W)$ and owning $V^o(W)$ for each income state, I cast the HJBVI (11) as a Linear Complementarity Problem and solve it. I then solve the KF equation as an eigenvalue problem using the transpose of the transition matrix used to solve the HJBVI, and check for equilibrium in the housing market. I update the guess for q using a bisection algorithm and re-solve the system until the price clears the market. See Appendix B for more details.

2.5 Calibration

I calibrate the model parameters partly to replicate key stylized facts as discussed in Davis and Van Nieuwerburgh (2015) and as well as to generate a wealth Gini coefficient that is close to that observed for most advanced economies. To this end, I study two economies in parallel, one with a wealth Gini of 0.7, and another of 0.6, which I label as high wealth gini (HWG) and low wealth gini (LWG) respectively. These levels capture the environment across several European

[17] This is a feature of the continuous time framework. In a discrete time framework optimality conditions are inequalities that only hold exactly when the constraint is binding.

[18] Saving has to be strictly positive when households hit the constraint, as dissaving reduces their wealth further, violating the constraint. Positive saving implies that $y_j + r\underline{W} \geq PC_j(\underline{W})$, $\forall j \in \{1, 2\}$. Furthermore, the optimality condition (12) holds everywhere, both within and on the boundary of the state space. Combining these two requirements yields the boundary condition (21).

[19] Although the grid is large, a solution is feasible as continuous time introduces a lot of sparsity and first order conditions hold with equality. As a result, a solution can be obtained in less than a minute on a personal machine. The high number of grid points results in smooth policy functions and distributions but does not materially increase accuracy. In the robustness checks reported below I use a smaller grid with 3,500 points since I only focus on comparing aggregate variables.

countries, as shown in Figure 1, and are comparable to values for advanced economies which are targeted in some of the studies listed above. The expenditure share for non-durable consumption goods α is set to 0.8, consistent with the discussion in Davis and Van Nieuwerburgh (2015). I set the risk aversion parameter σ to 1, as logarithmic preferences simplify the computation and improve the stability of the solution algorithm. The utility cost of renting ψ is set at 0.155, which delivers a homeownership rate of about 65%.

Next is the income process, which is the key source of heterogeneiety. The two income states $\{y_1, y_2\}$ do not represent states of unemployment and employment as is customary in the literature. Rather, the first income state y_1 represents income of a 'typical' household, whereas the second income state y_2 is a catch-all state for being hit by a rare but very good shock.[20] The corresponding Poisson intensities $\{\lambda_1, \lambda_2\}$ over these income states are set such that households spend most of the time in income state y_1, but with a low probability receive the high endowment.[21] Therefore, the calibration of these values does not follow estimated income processes as in the literature, but is set such that the model generates high wealth inequality as observed in the data. The flow probability λ_2, which denotes a jump to income state y_1, is set at 0.6 in the HWG economy, and at 0.1 in the LWG economy. The flow probability λ_1 is 0.05 in both economies. These imply that a given household is expected to be in the high income state only about 8% of the time in the HWG economy, and about 33% in the LWG economy. I normalize average income $\overline{y}$ to 1 as in Guerrieri and Lorenzoni (2017) and Fernández-Villaverde et al. (2019), and set the low income level $y_1 = 0.35$. This and the Poisson intensities imply a value for y_2 of 8.8 (HWG) and 2.3 (LWG) respectively. The implied income process then delivers a wealth Gini of 0.7 and 0.6 in the HWG and LWG economies, respectively. The discount rate ρ is set to 7.1%. Although higher than in the standard representative agent literature, it is common in models with even a minimum level of heterogeneity, such as two-agent models, to have some households in the economy which discount the future more heavily to generate wealth holdings that come close to the inequality observed in the data.[22]

The last set of parameters relate to the financial market. I fix the benchmark maximum LTV ratio θ at 90%, reflecting relatively loose borrowing constraints. This parameter will be the focus of the analysis. I set the interest rate at 2%. The fact that $r < \rho$ puts an upper bound on the wealth distribution, such that even high income households do not accumulate wealth above an upper threshold $\overline{W}$ (Huggett, 1993; Quadrini and Ríos-Rull, 1997). The wealth density is therefore bounded with support $[0, \overline{W}]$.[23] The parameter values, written in annual

[20] This is similar to Bayer et al. (2020) who include a low probability high income 'entrepreneur' state for households to generate high income and wealth distribution.

[21] A household expects to remain in state j with duration $1/\lambda_j$.

[22] See, for instance, Iacoviello and Pavan (2013), Guerrieri and Lorenzoni (2017) and Auclert et al. (2020). It is known that standard incomplete market models with limited sources of heterogeneity fail to generate a fat (Pareto) right tail, as observed in the data (Quadrini and Ríos-Rull, 1997). This is because households in the upper wealth percentiles are not driven by precautionary savings motives (Carroll, 1997). Heterogeneity in discount factors, amongst other factors, helps to generate a skewed distribution; see Krusell and Smith (1998), Krueger et al. (2016), Toda (2018) and Epper et al. (2020). The income process calibration in this paper generates a reasonable skew in the wealth distribution without the need for discount factor shocks, which add another state variable.

[23] Achdou et al. (2022) derive closed form expressions for the wealth distribution in an incomplete markets model with only idiosyncratic labour income risk, cast in continuous time. They show that the stationary

terms, are summarized in Table 1.

Table 1: Benchmark calibration

Parameter	**Value**
Discount rate (ρ)	0.071
Risk aversion (σ)	1
Utility cost of renting (ψ)	0.155
Non-durable consumption share (α)	0.8
Maximum LTV ratio (θ)	0.9
Risk-free interest rate (r)	0.02
Low income (y_1)	0.35
Poisson rate: low to high income (λ_1)	0.05
High Gini	
High income (y_2)	8.8
Poisson rate: high to low income (λ_2)	0.6
Low Gini	
High income (y_2)	2.3
Poisson rate: high to low income (λ_2)	0.1

3 Steady state analysis

I now describe household behaviour which is conditional on a low downpayment requirement, focusing on the HWG economy. I do not show the corresponding functions for the LWG economy when the results are quantitatively similar. Figure 2 shows the value functions associated with being a renter and an owner for the income endowment y_1. Since owning is prefered to renting, the value of being an owner is always greater or equal to that of being a renter. The point at which the two value functions are equal is the threshold cutoff point W_1^o. At this point, the solution satisfies equation (9) as an equality - the value matching condition - as well as a *smooth pasting* condition associated with stopping time problems (Dixit and Pindyck, 1994).[24] The solution for households on the high income state does not yield a corresponding threshold; given their high income flow, they find it optimal to always buy their house. As a result, the value function for owning weakly dominates that for renting over all wealth levels.

Figure 3 shows the policy functions for total expenditure, saving, and the allocation between non-durable consumption and housing services for households in income state y_1. In particular, it shows *two* sets of policies; one while renting to the left of the cutoff point W_1^o, and another while owning. Consumption of the non-durable good (c) and housing services (s) rise

density of the high income households is bounded both at the borrowing constraint $\underline{W}$ and also the right tail of the density ($\overline{W} < \infty$).

[24] Value matching implies that a household is indifferent between renting or owning exactly at the cutoff. Smooth pasting states that the derivatives of the value functions are also the same at the cutoff. As discussed in Achdou et al. (2022), smooth pasting is typically imposed as a boundary condition, but when a problem is posed as an HJB variational inequality it obtains as part of the solution (Øksendal, 1998).

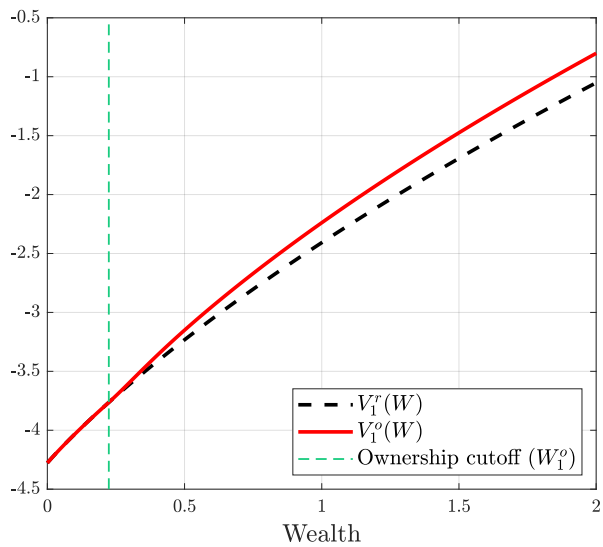

Figure 2: Value functions for income state y_1 (HWG)

Note: Wealth is a multiple of average income $\overline{y}$.

monotonically with wealth for renters, who rent a moderately-sized house. A common result in heterogeneous agent models is that households on the lower income endowment decumulate assets, whereas those on a high income endowment accumulate assets in good times.[25] Saving for households on y_1 however tends to zero as wealth approaches zero, with a corresponding dip in total expenditure, reflecting precautionary behaviour at low wealth levels and the boundary condition (21).

As wealth rises to the cutoff W_1^o, households switch to owning through a credit-constrained house purchase. The LTV constraint limits the size of the house that they can buy, and they switch to living in a house that is smaller than what they rent at wealth holdings just below W_1^o. This down-sizing reflects the fact that households get a higher utility from owning; and owning the smaller house more than compensates for the loss of higher housing services from renting. The point at which the house purchase becomes unconstrained, W_1^u, is also shown. Although households buy smaller houses in the interval $W \in [W_1^o, W_1^u)$ relative to what they were renting, they cut back on total expenditure to increase (reduce) their saving (dissaving). This captures an effort to remain an owner (that is, to have $W \geq W^o$) as well as to move off the collateral constraint (to have $W \geq W^u$). The behaviour of households in the LWG economy is very similar.

Households in the income state y_2 are owners throughout the wealth domain.[26] Their expenditure, as expected, is higher at all wealth levels, and they accumulate wealth through positive saving. High income households remain constrained at up to higher levels of wealth compared to those on a low income ($W_2^u > W_1^u$), since they have a higher demand for housing at each level of wealth compared to low income households. They have a high saving rate to move off

[25] See, for example, the discrete time analogue in Imrohoroğlu (1989, Figure A.1).

[26] I do not show their respective policy functions in the interest of space.

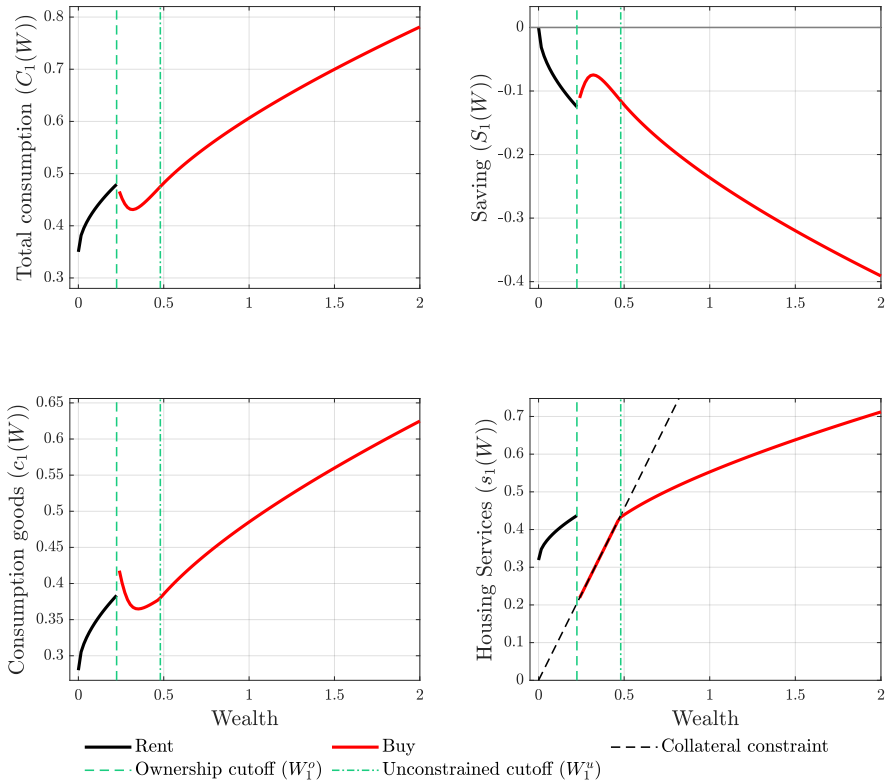

Figure 3: Policy functions for income state y_1 (HWG)

Note: Wealth is a multiple of average income $\overline{y}$.

the constraint and allocate expenditure optimally between consumption goods and housing.

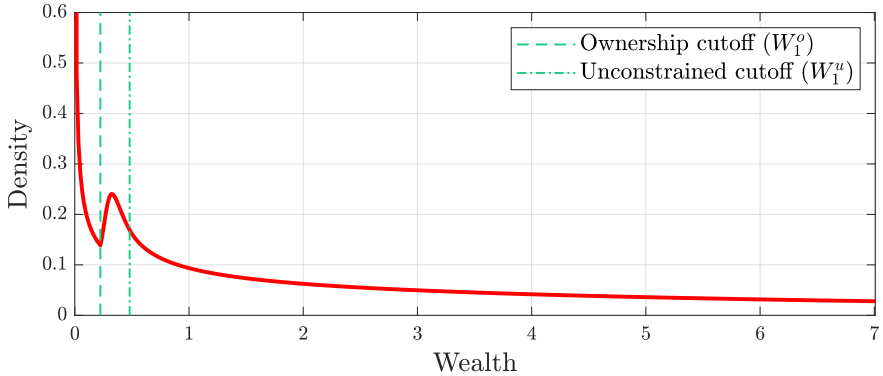

Figure 4: Unconditional density $g(W)$ (HWG)

Notes: The ownership and unconstrained cutoffs relate to households in the low income state. Wealth is a multiple of average income $\overline{y}$. The upper support of the density is close to 120 times average income.

Figure 4 plots the unconditional density for part of the domain on wealth, representing about 75% of all households in the HWG economy. As already mentioned, the Poisson flow rates for income in the HWG calibration imply that around 92% of households will belong to the density conditional on state y_1. Therefore, the unconditional density $g(W)$ mirrors the shape of $g_1(W)$. The density has a Dirac point mass at $W = 0$ and a fat right tail, bringing it close to empirical wealth distributions.[27] The Poisson intensities $\{\lambda_1, \lambda_2\}$ allow a few households to accumulate a relatively high level of wealth, generating significant wealth inequality despite the simplicity of the income process. The density on support $[0, W_1^o)$ represents the renters in the economy, in line with the stylized facts showing that renters tend to have little wealth. The region $[W_1^o, W_1^u)$ is populated by a density of constrained households in state y_1, where the rise in the density in this region reflects the accumulation of households driven by the change in saving behaviour. The share of LTV-constrained owners μ^{co} in this benchmark calibration is around 8%, whereas the share of renters and constrained owners μ^{rco} is about 40% (of which about 35% are renters). The model is also able to generate a sizable share of households (about 30% in both economies) who live hand-to-mouth, that is, households with zero wealth which consume all their endowment. This is close to the figure targeted in Kaplan and Violante (2014) and Kaplan et al. (2018).[28]

4 Comparative statics: tightening the borrowing limit

The results above are based on a relatively high LTV ratio $\theta = 0.9$ (90%) in the collateral constraint (3).[29] I now study the scenario where a policymaker cuts the LTV ratio by 10 percentage points and keeps it fixed indefinitely at 0.8. This could reflect the introduction of a macroprudential policy framework in which the regulator sets a minimum downpayment that is higher than what financial intermediaries ask for. Such a limit was imposed in New Zealand in 2013, where the LTV for most borrowers was and remains capped at 0.8 (Rogers, 2014). This policy action is known to all households, and is not expected to be revised in the future, so households do not face any uncertainty about future borrowing conditions. I therefore introduce the regime change as a so-called 'MIT' shock; a zero-probability event that cannot be anticipated and is not expected to hit again in the future. Upon being hit by the shock, the economy follows a deterministic adjustment path to the new stationary equilibrium at $\theta' < \theta$.

This policy affects the borrowing limit and housing demand through the collateral constraint. All else equal, a drop in the borrowing limit reduces demand for housing, at least by those households that are either on the limit or close to it. This exerts downward pressure on house prices, and causes unconstrained households to re-optimise, possibly increasing their holdings of housing. The net effect on house prices depends on the aggregate demand for housing. Figure

[27] A fraction of households in various countries have negative wealth, but the model is unable to capture this due to the state constraint $\underline{W} \geq 0$.

[28] These studies discuss poor and wealthy hand-to-mouth households, with the latter having zero liquid wealth but positive illiquid wealth. In this paper there is no such distinction since all wealth is liquid.

[29] Maximum LTV ratios vary across countries. In the euro area, they varied between 70% and 100% in 2016 (ECB, 2016, p.41). Furthermore, countries tend to have different LTV ratios depending on whether a household is a first time buyer or not, and in the case of buy-to-let, lower LTV ratios apply. See (Hallissey et al., 2014) for a discussion on different LTV limits for borrowers in Ireland.

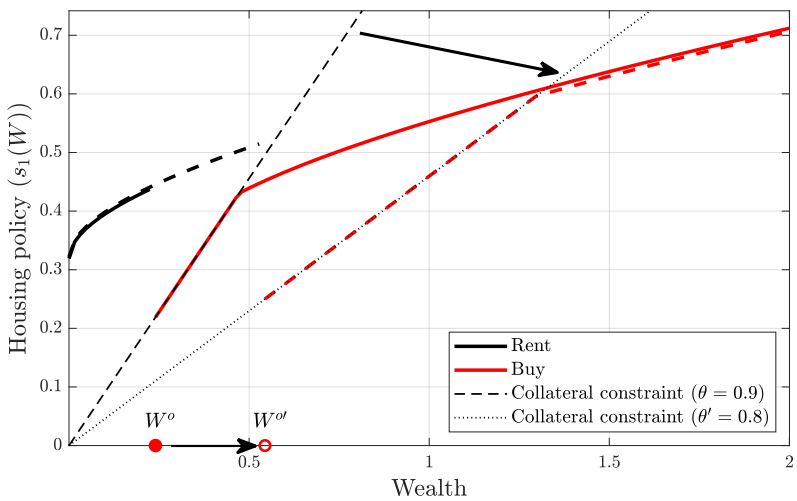

Figure 5: Housing policies before and after the LTV reduction for income state y_1 (HWG)

Notes: The solid lines are for an LTV ratio of 0.9, and the dashed lines for an LTV ratio of 0.8. The arrows show the clockwise rotation of the collateral constraint about the origin. The solid and empty dots represent the ownership cutoffs at the initial and terminal LTV ratios, respectively. Wealth is a multiple of average income $\overline{y}$.

5 shows the resulting change in behaviour as the LTV is lowered. The new ownership threshold for low income households $W_1^{o\,\prime}$ shifts up, and renters require more wealth to optimally switch to owning. The range over wealth across which the constraint binds is wider for all constrained households, irrespective of which income state they are in. Furthermore, households on the high income state have a slightly higher demand for housing services than before at wealth levels above the new unconstrained cutoff $W_2^{u\prime}$. Integrating over the entire wealth distribution, these changes lead to an aggregate drop in housing demand, which causes house prices to fall by about 0.8% in the HWG economy. Figure 6 shows that the aggregate demand curves are almost linear in the vicinity of the market equilibrium.

The story is the same in the LWG economy, but since this economy has a higher concentration of households on or close to the borrowing limit, the reduction in the LTV ratio induces a bigger drop in aggregate housing demand. Consequently, house prices in this economy drop by 3.4%, more than four times the drop in the HWG economy. This result affirms that the potential for distributive effects of macroprudential policy can hinge greatly on initial conditions.

Table 2 shows the changes in housing tenure, as well as aggregate measures including the wealth and housing wealth gini coefficients. At a lower borrowing limit a higher fraction of low income households rent, and the corresponding ownership rate falls from 65.2% to 59.4%. Moreover, the share of owners on the borrowing limit (μ^{co}) almost doubles from 7.8% to 14%. As a result, about 49% of households become either renters or constrained owners, up from 40%. Although a lower LTV ratio causes a significant proportion of the population to either rent or become constrained, it seems to have a negligible effect on wealth inequality as measured by the Gini coefficient. Figures 7 and 8 shed some light on why this is the case.

Figure 7 shows the wealth accumulation policy averaged over the two income states. The

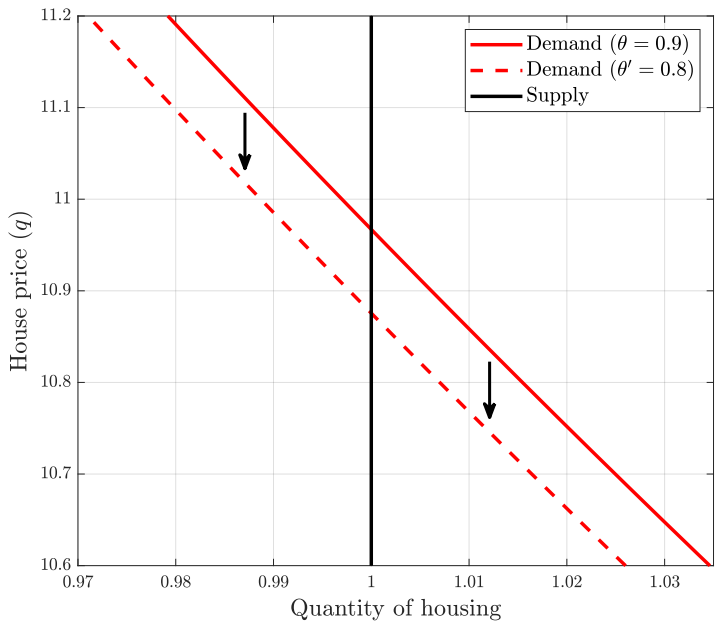

Figure 6: Comparative statics in the housing market (HWG)

Note: The x-axis denotes the quantity of housing, and supply is normalized to 1.

Table 2: Initial and terminal steady states

	High Wealth Gini			Low Wealth Gini		
	$\theta = 0.9$	$\theta' = 0.8$	Δ	$\theta = 0.9$	$\theta' = 0.8$	Δ
Household shares (%)						
Renters	34.8	40.6	5.8p.	34.4	40.0	5.6p.
Owners	65.2	59.4	-5.8p.	65.6	60.0	-5.6p.
Constrained owners (μ^{co})	7.8	14.0	6.2p.	11.8	27.2	15.4p.
Renters and constrained owners (μ^{rco})	39.9	48.9	9.0p.	42.2	56.3	14.1p.
Hand-to-mouth	30.0	31.9	2.0p.	29.6	31.2	1.6p.
Aggregates						
House price (q)	10.97	10.88	-0.8%	10.29	9.94	-3.4%
Leverage (%)	38.3	32.4	-5.9p.	49.2	42.8	-6.4p.
Wealth Gini	0.701	0.701	-0.1%	0.604	0.600	-0.6%
Housing Wealth Gini	0.582	0.609	4.6%	0.535	0.570	6.5%

Notes: Constrained owners is the fraction of owners that are on the collateral constraint. Leverage is calculated as the ratio of debt to housing wealth of owners. Δ denotes changes between the two LTV scenarios in percentage points (p.) or percentage rates (%). Numbers may not add up due to rounding.

LTV ratio tightening lowers saving at low levels of wealth but raises it at higher levels. This implies that there will be a relatively greater share of households at low wealth levels (since they save less and run down their wealth relatively faster), *and* a slightly greater density of households

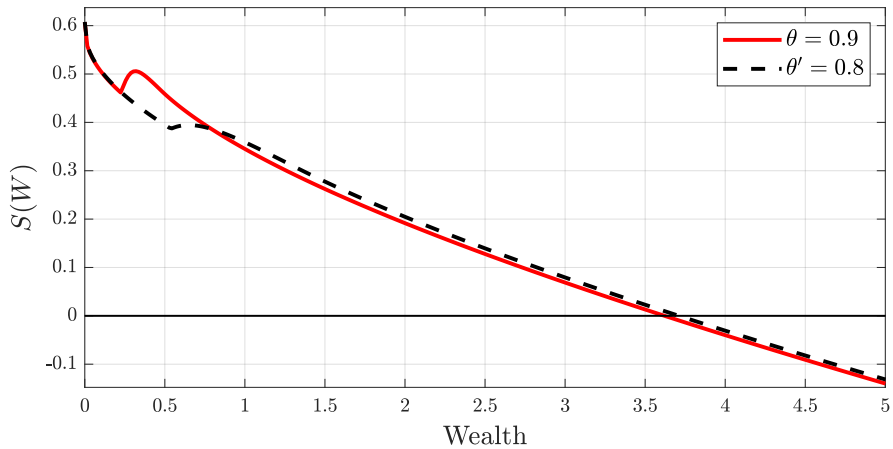

Figure 7: Saving policy functions averaged over y (HWG)

Note: Wealth is a multiple of average income $\overline{y}$.

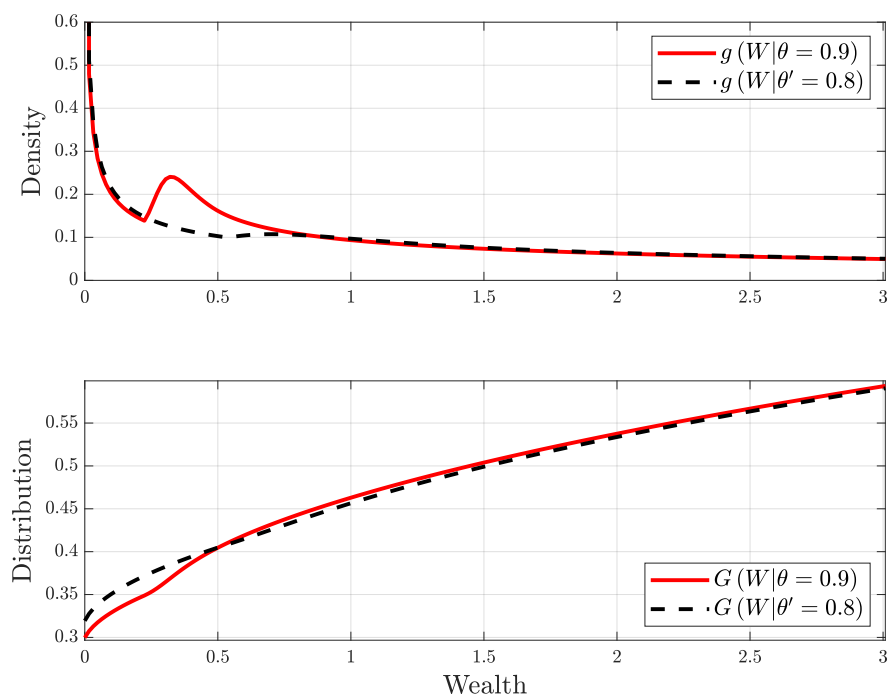

Figure 8: Unconditional wealth distributions for the two steady states (HWG)

Note: Wealth is a multiple of average income $\overline{y}$.

at higher wealth levels, since the latter are accumulating wealth at a slightly faster pace. The higher downpayment requirement causes the Lorenz curve to rotate slightly counter-clockwise about the fifty-fifth percentile of households, meaning that households below this percentile hold a slightly lower share of wealth than before, and households above this percentile hold a higher share. Figure 8 shows a larger share of households at very low wealth levels and as a result, the

new steady state preserves the area between the line of perfect equality and the Lorenz curve, keeping the wealth Gini coefficient virtually unchanged.

However, the macroprudential regime induces significant changes in the housing market. First, it alters the asset portfolio of the subset of households that is forced to switch to renting. Second, it limits the size of the house that can be purchased by a household that becomes constrained. Third, it also affects households that were and remain unconstrained, as it causes them to reduce their total consumption and save more, translating to a slight downsizing of their house.[30] Although overall wealth inequality is virtually unchanged as houses can be converted into bonds at no cost, the combined effect increases housing wealth inequality, raising the housing wealth gini by 4.6% in the HWG economy.[31] Effectively this concentrates housing wealth amongst the richer households.

The effect in the LWG economy is even greater, where the housing wealth Gini increases by 6.5%, despite the same terminal homeownership rates. This is because since households are relatively less heterogeneous in their wealth *ex-ante*, the policy affects a bigger share of households. Figure 9 shows the Lorenz curves for HWG economy, before and after the regime change.[32] The distributions of housing wealth at the high LTV *Lorenz dominates* those at the low LTV (Zoli, 2002). These clearly illustrate how, as discussed in Guerrieri and Lorenzoni (2017), policy affects most households along the distribution, not just those on the borrowing constraint or those who switch to renting. In sum, while a drop in the LTV ratio does not change the overall wealth Gini, it lowers homeownership and therefore increases housing wealth inequality. These effects are stronger the lower is the initial total wealth inequality.

[30] Admittedly, the last effect may be due to the lack of adjustment costs in the model, which would otherwise create a zone of inaction.

[31] These changes in housing wealth inequality are in line with the findings of Favilukis et al. (2017).

[32] The corresponding Lorenz curves for the LWG economy look similar, although in this economic the gap between the curves is slightly larger given the higher increase in the housing wealth gini.

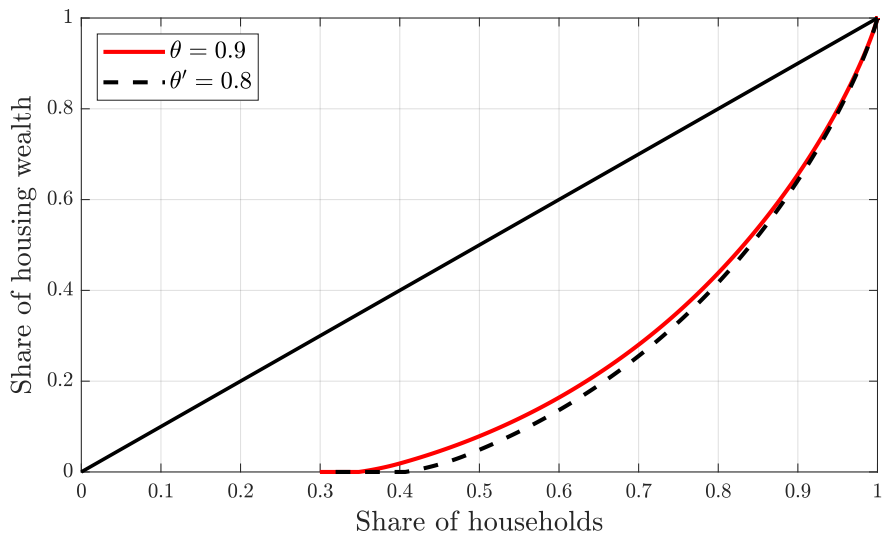

Figure 9: Lorenz curves for housing wealth (HWG)

Notes: The solid diagonal black line denotes the line of perfect equality. The curve for a given LTV ratio starts from the share of households that are hand-to-mouth and who have zero wealth, including no housing wealth. It rises above zero for households on and above the ownership cutoff.

5 Sensitivity and robustness checks

5.1 A range of LTV values

Are these findings sensitive to the level of the maximum LTV ratio? To answer this, I solve the model over the range of LTV values $\theta \in \Theta = [0.65, 0.9]$. A lower LTV ratio always leads to a fall in house prices and increases the ownership cutoff (and, as a result, the share of renters in the economy), as well as the share of constrained owners. In particular, the ownership cutoff increases by several factors over the level in the benchmark calibration, and at an LTV of 0.7, households on income state y_1 require four times as much wealth to find it optimal to switch to owning in both settings. Moreover, a few households on the high income state y_2 in the LWG economy also switch to renting as the LTV ratio falls below 0.8. As discussed in Sommer et al. (2013), there is a strong link between downpayment requirements and the homeownership rate, and LTV policies matter greatly for households that are on the borrowing limit. Figure 10 shows that while the homeownership rate falls by about the same irrespective of the initial shape of the wealth distribution, the dynamics in the share of constrained owners and equilibrium house prices differ a lot. Housing wealth inequality, although lower in the LWG economy, rises at a faster rate and approaches the level of the HWG economy at low LTV ratios.[33] As a result, the same policy can have different outcomes across countries with ex-ante different levels of wealth dispersion. This again emphasizes the point that the distribution can have a huge bearing on the outcome, and models which impose limited ex-ante heterogeneity miss these changing compositions by definition.

[33]Remember that total wealth inequality is virtually unchanged over the two steady states for each economy.

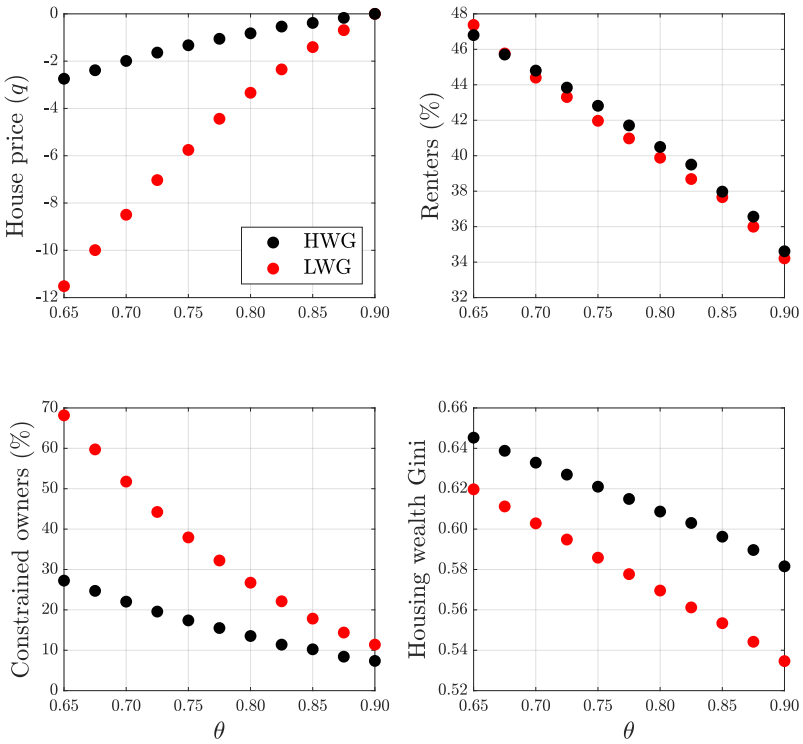

Figure 10: Comparative statics over a range of LTV values

Note: House prices are expressed in percentage deviation from their benchmark value at an LTV ratio of 0.9.

5.2 The long-run interest rate

The interest rate is exogenous at 2% in the analysis so far, but this parameter plays an important intertemporal role in the accumulation of wealth. I repeat the LTV ratio tightening in the context of a lower and higher long run interest rate of 1.5% and 4%, respectively. In the case of a lower interest rate I re-calibrate only the utility cost of renting ψ to 0.22 and 0.21 for the HWG and LWG economies, respectively so that the initial steady states in these lower interest rate economies are close to those in the benchmark economies. The steady state moments that I focus on are a wealth Gini of 0.7 and 0.6 respectively, and about 65% homeownership rate, at an LTV of 0.9. The re-calibration also delivers housing wealth Gini coefficients similar to the benchmark. Obtaining similar steady-state moments when the interest rate is 4% is harder and requires tweaking more parameters. I re-calibrate the Poisson flow rate into income state y_1, the utility cost of renting and the share of housing in total expenditure in both the HWG and LWG economies. The values $\{\lambda_2 = 0.38, \psi = 0.12, \alpha = 0.75\}$ in the HWG economy and $\{\lambda_2 = 0.093, \psi = 0.11, \alpha = 0.75\}$ in the LWG deliver steady state moments close to those in the benchmark economies.

As I conduct the comparative statics exercise I run into convergence problems when the interest rate is 1.5% and the LTV ratio falls below 0.75. Yet we can still observe differential long run impacts over the restricted LTV range $\theta \in \widetilde{\Theta} = [0.75, 0.9]$, where $\widetilde{\Theta} \subset \Theta$. In Figures 11

and 12 I show the mapping between aggregate house price and quantities at the two alternative interest rates respectively and θ, and also superimpose the corresponding mappings from the benchmark cases (hollow markers) for reference. At all interest rate levels, house prices fall as the LTV ratio is lowered and the share of constrained owners rises in both economies, relative to their values at an LTV of 0.9. Housing wealth inequality also rises. The main difference is the magnitude of changes. The lower the interest rate, the larger are the differences in steady-state house prices and the share of constrained owners. For instance, at an LTV of 0.75, house prices in the LWG economy are about 12.2% lower when r is 1.5%, compared to about -5.8% when r is 2% and -3% when the r is 4%. This pattern also extends to the share of renters in the economy and therefore the housing wealth Gini. For instance, in the LWG economy, the share of renters rises by 9, 8 and 6.5 percentage points over when r is 1.5%, 2% and 4%, respectively. Similarly, the housing wealth Gini rises by 0.062, 0.051 and 0.036 points, respectively as the LTV is reduced from 0.9 to 0.75.

Since some parameters are not set at the same values in these scenarios as in the benchmark case – especially when r is 4% – it is not totally innocuous to attribute the differential variation in house prices and share of constrained ownership over the LTV range wholly to different interest rates. Nevertheless, a higher interest rate induces households to save (dissave) relatively more (less), such that they become unconstrained at lower wealth levels and try harder to escape the zone at which they are constrained. Indeed, when r is 4% (1.5%), an LTV reduction causes a smaller (larger) increase in the share of constrained owners compared to when r is 2%. With a lower (higher) share of households on the borrowing constraint, a tightening of the LTV reduces housing demand relatively by less (more), and hence has a relatively weaker (stronger) effect on equilibrium house prices.

These additional findings are especially relevant in the context of a low natural interest rate environment caused by, *inter alia*, demographic or technological factors. The model suggests that using the LTV ratio as a macroprudential policy tool to meet financial stability objectives is likely to have stronger long-run effects on the housing market if long term interest rates are low. It follows that if the objective is to increase resilience to shocks by permanently reducing household leverage, then policymakers need to tighten the borrowing constraint by less relative to a scenario of high long-term interest rates.[34]

5.3 The initial homeownership rate

The homeownership rate in the benchmark scenario is calibrated to around 65%, which is the average discussed in Davis and Van Nieuwerburgh (2015) and Cloyne et al. (2019) for the US and UK and is the number most commonly used in the literature. Although the average rate across all 28 EU countries is also close to this level, there is significant heterogeneity in homeownership rates within the EU. For instance, in 2015 homeownership rates across the countries shown in Figure 1 were as low as 52% in Germany but exceeded 80% in Poland, Estonia, Malta and Latvia. Since the results may hinge on the *a priori* share of renters in the economy, I repeat the

[34]On the other hand, a low interest rate environment may be associated with a higher absolute level of household leverage, as is the case in this paper. The discussion here is on the absolute reduction in leverage; a smaller drop in the LTV ratio is needed to reduce leverage by 10 percentage points when interest rates are low.

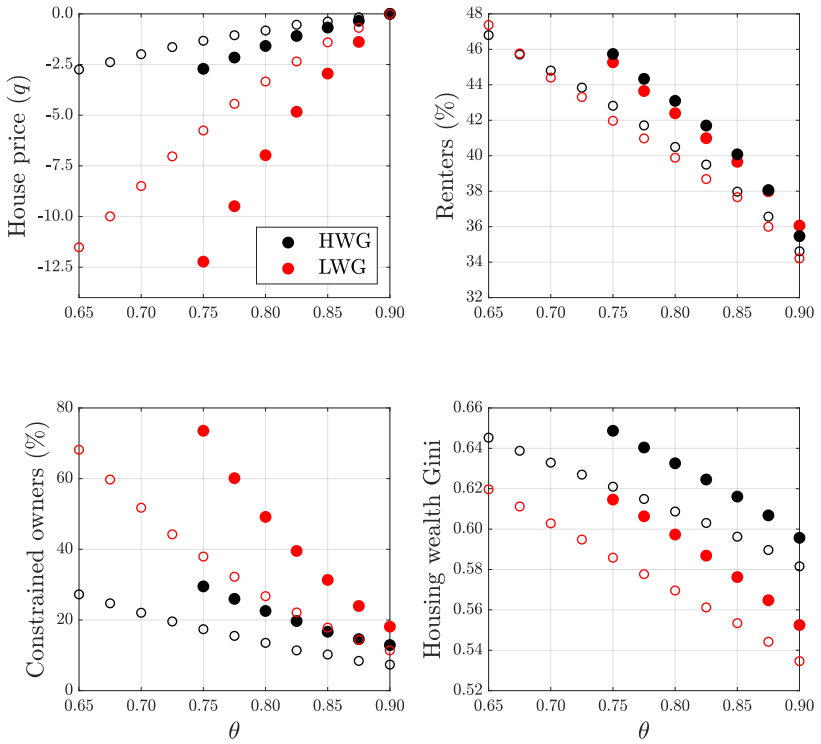

Figure 11: Comparative statics over a range of LTV values at $r = 1.5\%$

Notes: House prices are expressed in percentage deviation from their benchmark value at an LTV ratio of 0.9. The hollow markers denote the values from the benchmark calibration as shown in Figure 10. Aggregate quantities are only available for $\theta \in [0.75, 0.9]$.

main experiment with a higher initial homeownership rate close to 75% in both economies.[35] A reduction in the LTV from 0.9 to 0.8 causes about the same drop in homeownership rates in both economies, and causes a larger fall in house prices and increase in the share of constrained owners in the LWG economy (Table 3). There is also a stronger rise in housing wealth inequality in the LWG economy. These results are in line with those in the benchmark scenario.

[35] I am unable to increase homeownership by raising the utility cost of renting ψ higher than in the benchmark calibration as the numerical algorithm runs into convergence issues. I instead lower the discount rate ρ from 0.071 to 0.06 in both economies, and in the LWG economy, I reduce the share of housing expenditure α from 0.8 to 0.75. These produce the same homeownership rate of about 74% both economies.

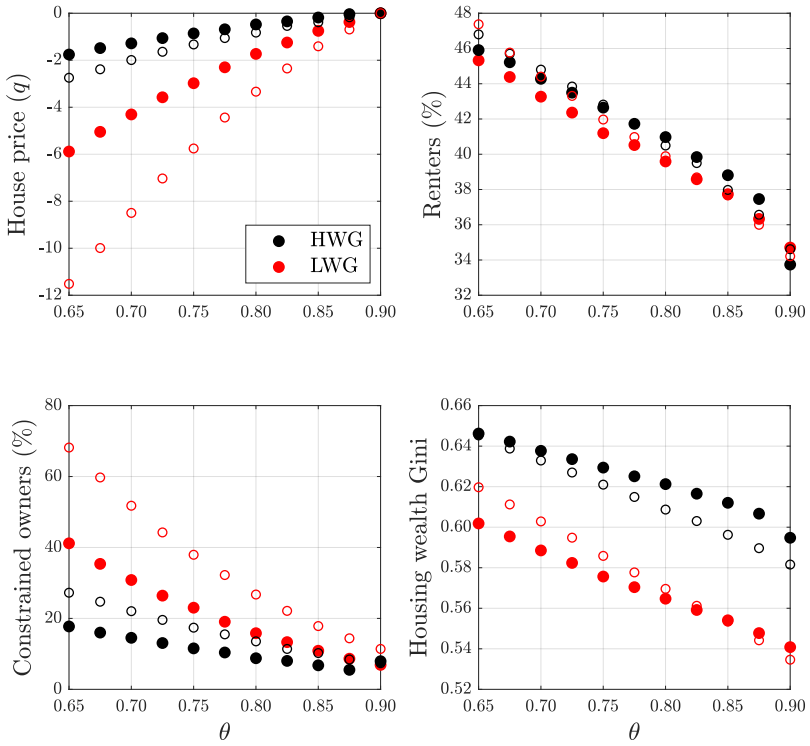

Figure 12: Comparative statics over a range of LTV values at $r = 4\%$

Notes: House prices are expressed in percentage deviation from their benchmark value at an LTV ratio of 0.9. The hollow markers denote the values from the benchmark calibration as shown in Figure 10.

Table 3: Initial and terminal steady states – higher initial homeownership

	High Wealth Gini			Low Wealth Gini		
	$\theta = 0.9$	$\theta' = 0.8$	Δ	$\theta = 0.9$	$\theta' = 0.8$	Δ
Household shares (%)						
Renters	26.2	34.1	7.9p.	26.5	34.7	8.3p.
Owners	73.8	65.9	-7.9p.	73.5	65.3	-8.3p.
Constrained owners (μ^{co})	6.3	10.6	4.4p.	12.2	25.5	13.3p.
Renters and constrained owners (μ^{rco})	30.8	41.1	10.3p.	35.5	51.4	16.0p.
Hand-to-mouth	23.1	27.6	4.4p.	22.8	26.8	3.9p.
Aggregates						
House price (q)	11.21	11.14	-0.6%	13.11	12.72	-3.0%
Leverage (%)	39.5	31.6	-7.9p.	55.1	46.1	-9.0p.
Wealth Gini	0.671	0.672	0.1%	0.577	0.575	-0.2%
Housing Wealth Gini	0.528	0.560	6.1%	0.488	0.533	9.2%

Notes: Constrained owners is the fraction of owners that are on the collateral constraint. Leverage is calculated as the ratio of debt to housing wealth of owners. Δ denotes changes between the two LTV scenarios in percentage points (p.) or percentage rates (%). Numbers may not add up due to rounding.

6 Conclusion

Are there distributive effects arising as unintended consequences of borrower-based macroprudential policy tightening? Does the extent of these effects hinge on the initial wealth distribution? This paper argues that the answers are 'yes' to both of these questions. A reduction in the LTV ratio tightens households' borrowing limit, causing some households which could just afford to buy a house against a mortgage to deleverage and switch to renting. It also causes households who were unconstrained before the policy to be up against their borrowing limit and reduce the size of their house. This causes an increase in housing wealth inequality, and wealthier households end up owning more of the housing stock ex-post. These effects are stronger when wealth is more equally distributed ex-ante, as in this case the policy has an effect on a greater share of households and the jump in housing inequality is higher. The same policy therefore has different outcomes across countries with different levels of wealth dispersion. Moreover, the effect of LTV policy on the housing market is inversely related to long-term interest rates, as a lower interest rate amplifies the drop in house prices and the rise in the share of constrained households, irrespective of the initial wealth inequality in the economy. Although homeownership rates vary significantly across Europe, these findings are robust to the initial homeownership rate and therefore are likely to apply to many of these countries.

This analysis does not internalize the benefits of a macroprudential framework. Asset price bubbles, default cascades, contagion, fire sales and capital flight are all unfavourable market phenomena with major implications for the real economy, and which policymakers aim to prevent by responding pre-emptively using the appropriate macroprudential tools. Nevertheless, policymakers should be aware that while such policies foster financial stability, they have the potential to alter the allocation of assets in an economy. The ability to target policy tools at the sector that is deemed to be at the heart of potential turmoil is an important effort in limiting these aggregate unintended consequences.

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Appendix A Key derivations

The main derivations presented in this appendix follow Sennewald and Wälde (2006) and Achdou et al. (2022).

A.1 Hamilton-Jacobi-Bellman Equation with Poisson uncertainty

I first derive the HJB equation from its counterpart in a discrete time version of the model for an owner, written in its stationary form and taking prices as given.[36] Income fluctuates between two states: $y_j \in \{y_1, y_2\}$. The problem is:

$$\max_{c,h} \sum_{t}^{\infty} u(c_t, h_t)$$
$$\text{s.t.} \quad b_{t+1} + q(h_{t+1} - h_t) = y_j + (1+r)b_t - c_t$$
$$-b_{t+1} \leq \theta q h_{t+1}.$$

Let $W_t = b_t + qh_t$ denote wealth; then the constraints can be written as:

$$W_{t+1} - W_t = y_i + rW_t - PC_t \tag{A.1}$$
$$W_{t+1} \geq (1-\theta)qh_{t+1} \tag{A.2}$$

where $PC_t = rqh_t + c_t$. The Bellman equation associated with this problem in the stationary equilibrium is:

$$V_j(W_t) = \max_{C} \; u(C_t) + \beta\left(\Pr(y_j)V_j(W_{t+1}) + (1-\Pr(y_j))V_{-j}(W_{t+1})\right)$$

subject to the constraints (A.1)–(A.2), and where C_t is the Cobb-Douglas bundle of non-durable goods and housing services, $\Pr(y_j)$ is shorthand notation to denote the conditional probability that households draw the same income y_j also in period $t+1$, $\Pr(y_{t+1} = y_j | y_t = y_j)$, and $-j = 2$ when $j = 1$ and vice versa. The Bellman equation and associated constraints in time steps of Δ are given by:

$$V_j(W_t) = \max_{C} \; \Delta u(C_t) + \beta(\Delta)\left(\Pr(\Delta, y_j)V_i(W_{t+\Delta}) + (1-\Pr(\Delta, y_j))V_{-j}(W_{t+\Delta})\right) \tag{A.3}$$
$$W_{t+\Delta} - W_t = \Delta(y_j + rW_t - PC_t) \tag{A.4}$$
$$W_{t+\Delta} \geq (1-\theta)qh_{t+\Delta}. \tag{A.5}$$

As the time steps Δ become small, the discount factor $\beta(\Delta)$ and income probability $\Pr(\Delta, y_j)$ can be approximated as $1 - \Delta\rho$ and $1 - \Delta\lambda_j$ respectively, where ρ is the rate of time preference and λ_j is the Poisson intensity parameter associated with draw y_j.[37] Plugging these in the

[36] See Sennewald and Wälde (2006) for the derivation of the HJB along a transition path.

[37] Recall that the density of a standard Poisson process is given by $\lambda e^{-\lambda t}$, and the probability that an event occurs before t is: $\int_0^t \lambda e^{-\lambda s} ds = [-e^{-\lambda s}]_0^t = 1 - e^{-\lambda t}$. In continuous time, λ is a flow probability.

Bellman equation and simplifying, we get:

$$V_j(W_t) = \max_C \ \Delta u(C_t) + (1-\Delta\rho)\left((1-\Delta\lambda_j)V_j(W_{t+\Delta}) + \Delta\lambda_j V_{-j}(W_{t+\Delta})\right). \tag{A.6}$$

Subtracting $(1-\Delta\rho)V_j(W_t)$ from both sides and dividing by Δ yields

$$\rho V_j(W_t) = \max_C \ u(C_t) + (1-\Delta\rho)\left(\frac{V_j(W_{t+\Delta}) - V_j(W_t)}{\Delta} - \lambda_j V_j(W_{t+\Delta}) + \lambda_j V_{-j}(W_{t+\Delta})\right). \tag{A.7}$$

Taking the limit $\Delta \to 0$, we have $\lim_{\Delta\to 0}(1-\Delta\rho) = 1$ and, by using the budget constraint (A.4), we can express the second limit as

$$\begin{aligned}
&\lim_{\Delta\to 0} \frac{V_j(W_{t+\Delta}) - V_j(W_t)}{\Delta} \\
&= \lim_{\Delta\to 0} \frac{V_j(W_t + \Delta(y_j + rW_t - PC_t)) - V_j(W_t)}{\Delta} \\
&= \frac{dV_j(W_t)}{dW}(y_j + rW_t - PC_t)
\end{aligned}$$

where the third line makes use of L'Hôpital's rule. Plugging this in (A.7) above and dropping time subscripts, we get the Hamilton-Jacobi-Bellman equations analogous to (8) and (10):

$$\rho V_j(W) = \max_C \ u(C) + \frac{dV_j(W)}{dW}(y_j + rW - PC_j) + \lambda_j(V_{-j}(W) - V_j(W)).$$

A.2 Kolmogorov Forward equation

The Kolmogorov Forward equation can be derived in a similar fashion to the HJB equation by referring to the discrete time counterpart in time steps of size Δ. Denote by $G_j(W,t)$ the distribution for households with wealth $W_t \leq W$ and the density as $\partial G_j(W,t)/\partial W = g_j(W,t)$. The fraction of households holding wealth up to W increases over time through dissaving ($S_j(W) < 0$)). For small Δ time steps, the evolution of wealth can be explained by $W_t = W_{t+\Delta} - \Delta S_j(W_{t+\Delta})$.[38] With only one income state, the fraction of households with wealth up to W is:

$$\Pr(W_{t+\Delta} \leq W) = \Pr(W_t \leq W - \Delta S(W_{t+\Delta}))$$

that is, the fraction of people who decumulate wealth enough to hold up to W. When income is stochastic, some households move out of the distribution for the income state j at the rate λ_j, whereas others move into this state at rate λ_{-j}. Therefore, the fraction of households with wealth lower than W evolves (increases) over time by:

$$\begin{aligned}
\Pr(W_{t+\Delta} \leq W|_{y_{t+\Delta}=y_j}) = &(1-\Delta\lambda_j)\Pr(W_t \leq W - \Delta S_j(W)|_{y_t=y_j}) \\
&+ \Delta\lambda_{-j}\Pr(W_t \leq W - \Delta S_{-j}(W)|_{y_t=y_{-j}})
\end{aligned}$$

$$\Rightarrow \ G_j(W, t+\Delta) = (1-\Delta\lambda_j)G_j(W - \Delta S_j(W), t) + \Delta\lambda_{-j}G_{-j}(W - \Delta S_{-j}(W), t).$$

[38] That is, instead of $W_t = W_{t+\Delta} - \Delta S_j(W_t)$, where the last term is saving at time t rather than $t+\Delta$.

Subtracting $G_j(W,t)$ from both sides and dividing throughout by Δ we have:

$$\begin{aligned}\frac{G_j(W,t+\Delta)-G_j(W,t)}{\Delta} =& \frac{G_j(W-\Delta S_j(W),t)-G_j(W,t)}{\Delta} \\ & -\lambda_j G_j(W-\Delta S_j(W),t)+\lambda_{-j}G_{-j}(W-\Delta S_{-j}(W),t)\end{aligned}$$

taking the limit $\Delta \to 0$ we get

$$\frac{\partial G_j(W,t)}{\partial t} = -S_j(W,t)\frac{\partial G_j(W,t)}{\partial W} - \lambda_j G_j(W,t) + \lambda_{-j}G_{-j}(W,t) \tag{A.8}$$

where the limit on the right makes use of l'Hôpital's rule:

$$\begin{aligned}&\lim_{\Delta\to 0} \frac{G_j(W-\Delta S_j(W),t)-G_j(W,t)}{\Delta} \\ &= \lim_{\Delta\to 0} \frac{\partial G_j(W-\Delta S_j(W),t)}{\partial W}(-S_j(W)) \\ &= -S_j(W)\frac{\partial G_j(W,t)}{\partial W}.\end{aligned}$$

Equation (A.8) is the law of motion in terms of the wealth distribution. Differentiating it with respect to wealth, and noting the definition for the wealth density above, we get the Kolmogorov Forward equation:

$$\frac{\partial g_j(W,t)}{\partial t} = \frac{\partial\left[-S_j(W,t)g_j(W,t)\right]}{\partial W} - \lambda_j g_j(W,t) + \lambda_{-j}g_{-j}(W,t).$$

In the stationary equilibrium the density is time-invariant and we have

$$0 = -\frac{d}{dW}\left[S_j(W)g_j(W)\right] - \lambda_j g_j(W) + \lambda_{-j}g_{-j}(W)$$

which is the equation shown in the text.

Appendix B Numerical algorithm

I describe the numerical procedure to solve the model in the benchmark calibration above, for the given calibration choice of logarithmic utility ($\sigma = 1$).[39] Algorithm 1 describes the procedure for obtaining the stationary solution which finds the policy functions, cut-off point W^o, conditional distributions $G_j(W)$ and market clearing prices $\{q, p\}$.

Algorithm 1 Solving for market clearing prices $\{q, p\}$

1. Discretize wealth W on a grid of I points between the bounds $\underline{W} = 0$ and some upper bound $\overline{W}$.
2. Set the parameter values and convergence tolerance limits for market clearing `tol-MC`.
3. Set a guess for the house price q, and compute the rental rate p using (4) and the aggregate price P using (15). Set upper and lower values for the guess of q, $\{\underline{q}, \overline{q}\}$, used by the bisection algorithm in the last step.
4. Calculate maximum feasible house ownership associated with the LTV constraint $h(W)^{\text{coll}} = \frac{W}{(1-\theta)q}$ at each point i on the wealth grid $W_{i \in I} \in [\underline{W}, \overline{W}]$.
5. Solve the renter's problem using a *upwind finite difference* scheme to solve the HJB (10), which yields $V_j(W)^r$ and the associated policy functions. See Algorithm 2 for further details.
6. Using $V_j(W)^r$ from the previous step, solve the HJBVI (11). This yields the value function $V_j(W)^o$, the associated policy functions and the cut-off W^o. See Algorithm 2 for further details.
7. Calculate the distribution $G_j(W)$ by solving the Kolmogorov-Forward equation (16), using the saving policy functions derived from step 6. See Algorithm 3 for more details.
8. Check for market clearing as in (20) by aggregating over all housing that is either owned or rented. If excess demand is less than or equal to `tol-MC` in absolute terms, stop. Otherwise, update q using a bisection algorithm and return to step 3 until convergence is achieved.

I use $I = 7{,}500$ equally-spaced points on the wealth grid for the benchmark model, and the value of $\overline{W}$ at which saving under the high income state is turns negative is just under 120 times average income $\overline{y}$. The entire stationary wealth density computed by the algorithm and the resulting policy functions and densities are very smooth at this fine discretization of the wealth state. Algorithm 2 describes the *upwind finite difference* scheme discussed in Achdou et al. (2022) to solve the HJB equation, and the added steps needed to solve the HJBVI.

[39] This choice simplifies some terms such as the boundary condition when wealth approaches zero and ensures that the algorithm is stable and converges to an (approximate) solution.

Algorithm 2 Solving the HJB equation or the HJBVI

1. For given parameters and prices, calculate steady state expenditure: $PC_j(W_i)^{ss} = y_j + rW_i$ for each income state y_j and calculate optimal steady state allocations for c^{ss} and s^{ss} as in (13) and (14).[40] Set a tolerance for the value function iteration `tol-VF`.

2. Use these values to initialise the value function as:

$$V_j(W_i)^0 = \frac{1}{\rho} \log \left(\left(\frac{c^{ss}}{\alpha} \right)^{\alpha} \left(\frac{(1 - \psi \mathbf{1}_{\text{rent}}) s^{ss}}{1 - \alpha} \right)^{1-\alpha} \right).$$

3. Approximate the derivative $dV_j(W)/dW$ using both a forward difference (F) and backward difference (B):

$$V_j'(W_i)^F \equiv \frac{V_j(W_{i+1}) - V_j(W_i)}{\Delta W}$$
$$V_j'(W_i)^B \equiv \frac{V_j(W_i) - V_j(W_{i-1})}{\Delta W}$$

   where ΔW denotes the distance between grid points.

4. Set the boundary condition (21) for households on income state y_1 at $W = \underline{W}$: $V_1'(W_1)^B = (y_1)^{-1}$.

5. Calculate $C_j(W)^k$, using (12) for each approximation of $V_j'(W_i)^k$, were $k \in \{F, B\}$. Calculate the resulting saving policies $S_j(W_i)^k$ for each k.

6. Choose between using the forward or backward difference on the basis of an *upwind* scheme:

$$V_j'(W_i) = \begin{cases} V_j'(W_i)^F \text{ if } S_j(W_i)^F > 0; & \Rightarrow \mathbf{1}_F = 1 \\ V_j'(W_i)^B \text{ if } S_j(W_i)^B < 0; & \Rightarrow \mathbf{1}_B = 1. \end{cases}$$

   This means that policy functions for total consumption and saving should be based on the forward difference approximation ($\mathbf{1}_F = 1$) when saving is positive, and on the backward difference approximation ($\mathbf{1}_B = 1$) when saving is negative.[41] At points where neither inequality is satisfied, a household is at the steady state ($S_j(W_i) = 0 \Rrightarrow \mathbf{1}_{ss} = 1$) and $C_j(W_i)^{ss}$ from step 1 applies.

7. Calculate the total consumption policy as: $C_j(W_i) = C_j(W_i)^F \mathbf{1}_F + C_j(W_i)^B \mathbf{1}_B + C_j(W_i)^{ss} \mathbf{1}_{ss}$, the implied policies $c_j(W_i)$ and $s_j(W_i)$ and the associated utility $u_j\left(c_j(W_i), s_j(W_i)\right)$.

[40] In the renter's problem these are the unconstrained allocation shares α and $1 - \alpha$ of total expenditure respectively.

[41] Since the value function is concave, the case where both $S_j(W_i)^F > 0$ and $S_j(W_i)^B < 0$ is ruled out.

8. (a) HJB: solve for the value function using the iterative scheme:

$$\begin{aligned}
&\frac{V_j(W_i)^{n+1} - V_j(W_i)^n}{\Delta} + \rho V_j(W_i)^{n+1} \\
&\quad = u\left(c_j(W_i)^n, s_j(W_i)^n\right) \\
&\quad + V_j'(W_i)^{n+1,F} S_j(W_i)^{n,F} \mathbf{1}_F + V_j'(W_i)^{n+1,B} S_j(W_i)^{n,B} \mathbf{1}_B \\
&\quad\quad + \lambda_j \left(V_{-j}(W_i)^{n+1} - V_j(W_i)^{n+1}\right)
\end{aligned}$$

which is implicit in the value function and where n is the iteration counter and Δ is a step size, set to a large number. Stack over j the value functions $V_j(W)^n$, $V_j(W)^{n+1}$ and utility $u_j\left(c_j(W_i)^n, s_j(W_i)^n\right)$ into the $jI \times 1$ vectors $\mathbf{V}^n$,$\mathbf{V}^{n+1}$, and $\mathbf{u}^n$ respectively, and collect all other right-hand side elements into the $jI \times jI$ transition matrix $\mathbf{A}^n$ as:

$$\mathbf{A}^n = \begin{bmatrix}
\chi_{1,1} & \omega_{1,1} & 0 & \dots & 0 & \lambda_1 & 0 & 0 & \dots & 0 \\
\zeta_{2,1} & \chi_{2,1} & \omega_{2,1} & 0 & \dots & 0 & \lambda_1 & 0 & \dots & 0 \\
0 & \zeta_{3,1} & \chi_{3,1} & \omega_{3,1} & 0 & \dots & 0 & \lambda_1 & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & 0 & \zeta_{I,1} & \chi_{I,1} & 0 & 0 & \dots & 0 & \lambda_1 \\
\lambda_2 & 0 & \dots & 0 & 0 & \chi_{1,2} & \omega_{1,2} & 0 & \dots & 0 \\
0 & \lambda_2 & 0 & \dots & 0 & \zeta_{2,2} & \chi_{2,2} & \omega_{2,2} & \dots & 0 \\
0 & 0 & \lambda_2 & 0 & \dots & 0 & \zeta_{3,2} & \chi_{3,2} & \omega_{3,2} & \dots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \lambda_2 & 0 & \dots & 0 & \zeta_{I,2} & \chi_{I,2}
\end{bmatrix}$$

where, given the definitions of the forward and backward difference approximations

$$\begin{aligned}
\zeta_{i,j} &= -\frac{S_j(W_i)^{n,B}}{\Delta W}\mathbf{1}_B \\
\chi_{i,j} &= -\frac{S_j(W_i)^{n,F}}{\Delta W}\mathbf{1}_F + \frac{S_j(W_i)^{n,B}}{\Delta W}\mathbf{1}_B - \lambda_j \\
\omega_{i,j} &= \frac{S_j(W_i)^{n,F}}{\Delta W}\mathbf{1}_F.
\end{aligned}$$

Note that $\zeta_{i,j} = \omega_{I,j} = 0$. The iterative scheme can then be written in matrix form as:

$$\frac{1}{\Delta}(\mathbf{V}^{n+1} - \mathbf{V}^n) + \rho \mathbf{V}^{n+1} = \mathbf{u}^n + \mathbf{A}^n \mathbf{V}^{n+1}.$$

The new value function iterate $\mathbf{V}^{n+1}$ is obtained as:[42]

$$\mathbf{V}^{n+1} = \left[\left(\frac{1}{\Delta} + \rho\right)\mathbf{I} - \mathbf{A}^n\right]^{-1}\left[\mathbf{u}^n + \frac{1}{\Delta}\mathbf{V}^n\right].$$

If the maximum absolute difference between $\mathbf{V}^n$ and $\mathbf{V}^{n+1}$ is less than `tol-VF`, stop. Otherwise, go to step 3.

[42] The A matrix is large but sparse, but the system can still be solved using sparse matrix routines.

(b) HJBVI: recall the problem (11). Using the same notation as in 8(a), this can be written as:

$$\min\{\rho\mathbf{V} - \mathbf{u} - \mathbf{A}\mathbf{V}, \mathbf{V} - \mathbf{V}^r\} = 0$$

where the same matrix defintions apply, and $\mathbf{V}$,$\mathbf{u}$ and $\mathbf{A}$ relate to the owner's problem are obtained in the same way following steps 1-8(a), while $\mathbf{V}^r$ is from the renter's problem obtained following steps 1–8(a). Denote by $\mathbf{B}$ the matrix

$$\mathbf{B} = \left[\left(\frac{1}{\Delta} + \rho\right)\mathbf{I} - \mathbf{A}^n\right]$$

by $\mathbf{z}$ the slack vector

$$\mathbf{z} = \mathbf{V} - \mathbf{V}^r$$

and by $\mathbf{q}$ the matrix:

$$\mathbf{q} = -\mathbf{u} - \frac{1}{\Delta}\mathbf{V} + \mathbf{B}\mathbf{V}^r.$$

Then the problem can be solved as a Linear Complementarity Problem:

$$\begin{aligned} \mathbf{z}^T(\mathbf{B}\mathbf{z} + \mathbf{q}) &= 0 \\ \mathbf{z} &\geq 0 \\ \mathbf{B}\mathbf{z} + \mathbf{q} &\geq 0 \end{aligned}$$

and solved using a suitable LCP solver, which yields the value function for owners as the vector $\mathbf{V}$, the associated policy functions and the ownership cutoff which occurs when $\mathbf{V} \neq \mathbf{V}^r$.[43] If the maximum absolute difference between successive iterates of $\mathbf{V}$ is less than `tol-VF`, stop. Otherwise, go to step 3.

As discussed in Achdou et al. (2022), the upwind scheme strategy satisfies the Barles-Souganidis conditions such that the approximation converges to the solution of the HJB (Barles and Souganidis, 1991). In order to satisfy the borrowing constraint, the derivative at the lower end of the state space is calculated by setting (21) as an equality and applying it when the drift is negative, that is, when agents are approaching the borrowing constraint from above. Therefore, the boundary condition in step 4 only applies to households on the low income state, and relates only to the backward difference. See Achdou et al. (2022) for further details.

[43] I use the LCP solver by Yuval Tassa, available at https://www.mathworks.com/matlabcentral/fileexchange/20952-lcp-mcp-solver-newton-based (accessed 3 December 2019).

Algorithm 3 Solving the Kolmogorov-Forward equation

1. The stationary KF equation (16) can be discretized as:

$$0 = -\left[\frac{S_j(W_i)^{n,F} g_j(W_i) - S_j(W_{i-1})^{n,F} g_j(W_{i-1})}{\Delta W}\mathbf{1}_F + \frac{S_j(W_{i+1})^{n,B} g_{i+1,j} - S_j(W_i)^{n,B} g_j(W_i)}{\Delta W}\mathbf{1}_B\right]$$
$$- g_j(W_i)\lambda_j + g_{-j}(W_i)\lambda_{-j}$$

   which makes use of the same upwind scheme as above. Collecting like terms, we have:

$$0 = g_j(W_{i-1})\frac{S_j(W_{i-1})^{n,F}}{\Delta W}\mathbf{1}_F + g_j(W_i)\left(-\frac{S_j(W_i)^{n,F}}{\Delta W}\mathbf{1}_F + \frac{S_j(W_i)^{n,B}}{\Delta W}\mathbf{1}_F - \lambda_j\right)$$
$$- g_j(W_{i+1})\frac{S_j(W_i)^{n,B}}{\Delta W}\mathbf{1}_B + g_{-j}(W_i)\lambda_{-j}.$$

   This can be written in matrix form as:

$$\mathbf{0} = \mathbf{A}^{\mathrm{T}}\mathbf{g}$$

   where $\mathbf{A}^{\mathrm{T}}$ is the transpose of $\mathbf{A}$ from the last iteration of Algorithm 2 and

$$\mathbf{g} = [g_1(W_1)\ \ldots\ g_1(W_I)\ ,\ g_2(W_1)\ \ldots\ g_2(W_I)]^{\mathrm{T}}.$$

   This is an eigenvalue problem with the constraint (17) that the entire density must integrate to 1. To solve it, create a vector $\widetilde{\mathbf{b}}$ of zeros of length $jI \times 1$ and set an element to a small number, and set all the entries of the corresponding row in matrix $\mathbf{A}^{\mathrm{T}}$ to 0 except for 1 in the diagonal, and denote this matrix as $\widetilde{\mathbf{A}}^{\mathrm{T}}$.[44]

2. Obtain the density vector $\widetilde{\mathbf{g}}$ as:

$$\widetilde{\mathbf{g}} = \left[\widetilde{\mathbf{A}}^{\mathrm{T}}\right]^{-1}\widetilde{\mathbf{b}}.$$

3. Obtain the normalised stacked density vector $\mathbf{g}$ as:

$$\mathbf{g} = \frac{\widetilde{\mathbf{g}}}{\sum \widetilde{\mathbf{g}}\Delta W}.$$

4. Obtain the discretized conditional distributions $G_j(W)$, $\mathbf{G_1}$ and $\mathbf{G_2}$, as the cumulative

[44] This method, discussed in Achdou et al. (2022), is one way of solving the eigenvalue problem that satisfies (17). Without this adjustment, the matrix $\mathbf{A}^{\mathrm{T}}$ is singular and the next steps do not work. Although an eigenvalue routine such as the `eigs` function in MATLAB can also be used, it runs into numerical problems when I is higher than about 1,300 in this example.

sum of the densities weighed by the step size ΔW:

$$\mathbf{G}_{1,i} = \sum_{1}^{I} \mathbf{g} \Delta W$$
$$\mathbf{G}_{2,i} = \sum_{I+1}^{2I} \mathbf{g} \Delta W.$$